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LAPLACIAN GROWTH OF PARALLEL NEEDLES: THEIR MULLINS-SEKERKA INSTABILITY

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We study analytically the kinetics of growth of parallel needles, using a conformal transformation to set up the iterative nonlinear equations. This allows to build the discrete Fokker-Planck equation for the probability of finding at time t a given distribution of needle lengths. We consider here two specific cases: we find the exact Fokker-Planck equation for pairs of needles and its solutions, and the linear behavior of a set of n needles with equal initial lengths. The corresponding Fokker-Planck equations show the short-wavelength Mullins-Sekerka instability of these parallel needles, and the possible structure of the screening leading to the scale invariance of the model.

1 Introduction

Laplacian growth is an ubiquitous process of considerable physical interest: start of dendritic growth of crystalline structures during solidification when the diffusion length is large, electrodeposition, viscous fingering, dielectric breakdown or growth of bacterial colonies, belong to Laplacian growth and can be described by Diffusion Limited Aggregation models (DLA)[1]. Related physical phenomena include mechanical cracking, like mud during drying, crack formations in pieces of materials under strain. The Laplacian field takes its origin in the diffusion of material in front of the growing structures, particles which aggregate, nutrient in the growth of bacterial colonies, or in the electric field as in electrodeposition or in dielectric breakdown. In most situations, branching occurs during the growth, leading to DLA structures. In spite of the apparent simplicity of the physical laws governing these systems, and the very simple simulation models containing the essence of the physics, our analytic understanding of DLA remains very unsatisfactory. Essence studies have used extended numerical simulations to extract the scaling laws governing these systems. Therefore a more modest but useful approach consists in a better understanding of DLA growth in the absence of branching[2,3]. Again, in this simpler problem of growing needles, we can consider radial or parallel needles, as well as reflection (*model R*) or absorption (*model A*[4]) of particles (see for instance Krug[5], for a review). In this paper we will concentrate our attention on systems of parallel absorbing needles which, we think, are of more fundamental interest. In two dimensions, conformal mapping allows to obtain analytical results as shown by Derrida and Hakim[6] for radial needles.

An important question in such systems is the competition between the needles. The instability of an initial distribution of equal lengths needles, as well as unstable modes of a growing flat interface in dendritic growth, is known as the "Mullins-Sekerka" instability[7]. In a recent experiment, Losert *et al.*[8] showed the spatial period-doubling instability of dendritic arrays in directional solidification as suggested theoretically by Warren and Langer[9]. It will be the purpose of this

preliminary paper to establish the basic Fokker-Planck equation for the growth of n parallel needles, in the linear regime, which leads to the Mullins-Sekerka behavior. We will indicate in a few words, on the basis of the exact solution for pairs of needles, what we can expect for the general behavior of n parallel needles.

To study analytically the needle models of Laplacian growth in 2-dimensional systems, the most convenient approach is to use conformal transformations. We will only give in Sec. 2, the main arguments to set up these transformations, and the reader is referred to Shraiman and Bensimon[10], Szép and Lugosi[11], Peterson and Furry[12], Kurtze[13], and Derrida and Hakim[6]. For the dynamics, we will follow here Derrida *et al.*[6,14]. In Sec. 3 we write the closed set of equations which describes the time evolution in a compact matrix notation. In Sec. 4, we solve exactly the general two-needles case, and establish the associated Fokker-Planck equation. The case of n growing parallel needles is examined in Sec. 5, and the linearized equations are derived in Sec. 6. These Fokker-Planck equations reveal the short wavelength instability leading to period-doubling.

Two ingredients are needed for the Laplacian growth : on the one hand the Laplacian behavior determines the long range interaction characteristic of DLA growth, but on the other hand the inherent Mullins-Sekerka instability could not operate without the presence of local noise. This noise can be introduced [6,14] in the initial state. We chose here to consider a discrete model with diffusing particles of finite size $\delta\ell$ sticking to the needles at discrete time intervals.

2 The basic conformal transformations

The classical way to parametrize in a convenient manner Laplace's equation with a zero potential boundary condition on a set of n parallel needles is to introduce first a transformation which maps the unit circle in the complex plane z onto an n-branched star in the complex plane ω:

$$\omega = f(z) = Az\prod_{j=0}^{n-1}\left(1 - e^{i\theta_j}/z\right)^{\alpha_j} \quad with \quad \sum_{j=0}^{n-1}\alpha_j = 2. \tag{1}$$

In this transformation, $\pi\alpha_j$ is the angle between two successive needles $\{j-1, j\}$, the sum of the angles being 2π (equ.(1)). The angles θ_j fix the lengths ℓ_i of the needles, and A is a parameter which will be determined below.

A second transformation maps the star in the complex plane ω into a set of parallel needles in a complex plane Ω :

$$\Omega = \log\omega. \tag{2}$$

In the plane Ω, the Laplacian field is $\Phi(\Omega) = Re\left[\log\left(f^{-1}(\exp\Omega)\right)\right]$.

The tip positions in plane z are parametrized by the angles ϕ_i :

$$z_i = \exp(i\phi_i)\ ,\ 0 \le i \le n-1\ , \tag{3}$$

and the lengths of the needles are the modulus of $\log(f(z))$ at the points $z = z_i$:

$$\ell_i = \log(4A) + \sum_{j=0}^{n-1}\alpha_j\log\left|\sin\left((\phi_i - \theta_j)/2\right)\right|\ ,\forall\, 0 \le i \le n-1\,. \tag{4}$$

The n additional constraints must be imposed onto the angles ϕ_i and θ_j which take into account the fact that the needle tips maximize $|f(z)|$ at $z = z_i$:

$$\sum_{j=0}^{n-1} \alpha_j \cot((\phi_i - \theta_j)/2) = 0 \,, \forall\, 0 \le i \le n-1 \,. \tag{5}$$

Now for the kinetics, the growth rate is supposed to be proportional to the potential gradient along the needles. In addition, the growth is supposed to be restricted to the tips. Therefore, following refs.[6,14], where only the tips grow, while the needles remain at zero potential (*model A*), the growth rate of the needles is:

$$\frac{d\ell_i}{dt} \propto \left[\sum_{j=0}^{n-1} \alpha_j \left(1 + \cot^2((\phi_i - \theta_j)/2)\right)\right]^{-1/2} \,, \forall\, 0 \le i \le n-1 \,. \tag{6}$$

This equation will be used to determine the growth probability of the needles.

Now we have all the ingredients necessary to build the recursion relations to describe the evolution of our system.

3 Matrix form of the basic equations

It appears first very convenient to introduce the matrix $\mathbf{C} = \{c_{ij}\}_{0 \le i,j \le n-1}$ where:

$$c_{ij} = \cot((\phi_i - \theta_j)/2) \tag{7}$$

and condition (5) may be simply written ($\overrightarrow{\alpha}$ is the vector $\{\alpha_i\}_{0 \le i \le n-1}$):

$$\mathbf{C}.\overrightarrow{\alpha} = \overrightarrow{0} \,. \tag{8}$$

In a similar way, we can define $\mathbf{D} = \{d_{ij}\}_{0 \le i,j \le n-1}$ where $d_{ij} = 1 + c_{ij}^2$. Then condition (6) is :

$$d\ell_i/dt \propto \{\mathbf{D}.\overrightarrow{\alpha}\}_i^{-1/2}. \tag{9}$$

3.1 Infinitesimal growth

We consider now that between t and $t + \delta t$, a particle of size $\delta\ell$ sticks on needle i. This increase of length is supposed much smaller than the distance between needles, $\pi\alpha_j$. The equations can be linearized: for a variation $\delta\ell_i$ of the length ℓ_i corresponds the variations $\delta\phi_i$ of ϕ_i and $\delta\theta_i$ of θ_i. From (4) and (5) we find:

$$\delta\ell_i = \delta A/A - \sum_{j=0}^{n-1} (\alpha_j/2) \cot((\phi_i - \theta_j)/2)\, \delta\theta_j \,. \tag{10}$$

As there are $(2n+1)$ variables in our system of $2n$ equations we have to impose a supplementary constraint. One choice has been to fix one angle (for example $\theta_0 = 0$, ref. [14]). Here we will make the mean of the ℓ_i to remain zero (this choice does not single out any angle). This fixes A via equation (4). Using (10), and $\sum_i \delta\ell_i = 0$,

$$(\delta A/A)\mathbf{1} = 1/(2n)\, \mathbf{O}.\mathbf{C}.\overrightarrow{\alpha\delta\theta} \,, \tag{11}$$

where $\overrightarrow{\alpha\delta\theta} = \{\alpha_i \delta\theta_i\}_{0 \leq i \leq n-1}$. Here, $\mathbf{1}$ is the unit matrix and $\mathbf{O} = \{1\}_{ij}$ a matrix filled of 1's. Replacing (11) into (10) gives $\delta\ell_i$ as functions of $\delta\theta_j$,

$$\overrightarrow{\delta L} = -1/2 \ (\mathbf{1} - (1/n)\ \mathbf{O}) . \mathbf{C} . \overrightarrow{\alpha\delta\theta} \,, \tag{12}$$

where $(\mathbf{1} - (1/n)\ \mathbf{O})$ is a projector on the subspace orthogonal to the completely symmetrical component (for a vector $\overrightarrow{V}$ it is $\widetilde{V}_0 = (1/n)\sum_i v_i$). The choice, $\overrightarrow{\delta L}[j] = {}^t\{-\frac{\delta\ell}{n}, -\frac{\delta\ell}{n}, ..., \{\delta\ell - \frac{\delta\ell}{n}\}_j, ..., -\frac{\delta\ell}{n}\}$ corresponding to adding a particle $\delta\ell$ on needle j, ensures the conservation of the mean length (equal to zero). The ^{t} is for transposition (column vector). Eq. (12) allows to express the variations of $\delta\theta_j$ as functions of the $\overrightarrow{\delta L}$. We can also relate $\delta\phi_i$ to $\delta\theta_j$; Eq. (5) gives :

$$0 = \sum_{j=0}^{n-1} \alpha_j \left(\delta\phi_i - \delta\theta_j\right) \left(1 + \cot^2\left((\phi_i - \theta_j)/2\right)\right) \ . \tag{13}$$

This makes it possible now to calculate the variations $\delta\phi_i$:

$$\delta\phi_i = (\mathbf{D}.\overrightarrow{\alpha\delta\theta})_i / (\mathbf{D}.\overrightarrow{\alpha})_i \ . \tag{14}$$

Now, Eq. (9) which provides the increase of needle length per unit time in the diffusion field, enables us to introduce the growth probability p_i for a particle of size $\delta\ell$ to stick on the tip of needle i during a time δt. Let $(\mathbf{D}.\overrightarrow{\alpha}) = \overrightarrow{\Pi}$,

$$p_i = d\ell_i / (\sum_i d\ell_i) \ = (\Pi_i)^{-1/2} \,/\, (\sum_i (\Pi_i)^{-1/2}). \tag{15}$$

The set of equations, (12), (14) and (15) together with (8) define the successive iterations of the growth: if at time t, all the parameters are known, (15) determines the growth probabilities on the tips; knowing the $\delta\ell_i$ we calculate $\overrightarrow{\delta\theta}$ via (12) then $\overrightarrow{\delta\varphi}$ via (14), and then the perturbation $\delta\mathbf{C}$ of $\mathbf{C}$, and $\delta\mathbf{D}$ of $\mathbf{D}$. To calculate $\overrightarrow{\delta\theta}$ as a function of $\overrightarrow{\delta L}$ we have to invert Eq. (12). Unfortunately, relation (8) shows that $\overrightarrow{\alpha}$ is an eigenvector of $\mathbf{C}$ for the eigenvalue 0, and therefore $\mathbf{C}$ cannot be inverted directly.

3.2 Invariants

In a global rotation $\delta\vartheta$ of the angles θ et ϕ the configuration of the needles and thus all the preceding expressions remain invariant. For example, $\mathbf{C}$ is left unchanged and from (12) and (8) $\overrightarrow{\delta L}$ also remains invariant:

$$\overrightarrow{\delta L} = -(1/n)\ (\mathbf{1} - (1/n)\ \mathbf{O}) . \mathbf{C} . \left(\overrightarrow{\alpha\delta\theta} + \delta\vartheta\, \overrightarrow{\alpha}\right) = -(1/n)\ (\mathbf{C} - (1/n)\ \mathbf{O}.\mathbf{C}) . \overrightarrow{\alpha\delta\theta} \,. \tag{16}$$

4 The two-needles problem

We are now ready to solve the two needles case. Here $n = 2$, but the needles are periodically repeated due to the cyclic boundary conditions imposed as shown in Fig.1. With the zero mean length constraint, we write,

$$\widetilde{\ell}_0 = (\ell_0 + \ell_1)/2 \equiv 0; \ \ \widetilde{\ell}_1 = (\ell_0 - \ell_1)/2.$$

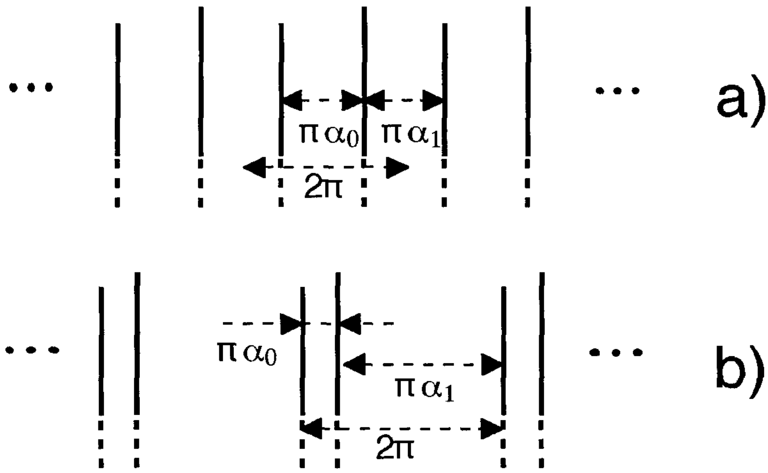

Figure 1. a) Set of equally distant pairs of needles ($\alpha_0 = \alpha_1 = 1$); b) General case.

The problem is completely defined by the knowledge of the probability $P(\widetilde{\ell}_1, t)$ to find at time t a length gap $\widetilde{\ell}_1$. The general form of $\mathbf{C}$, due to condition (8) and of $\mathbf{D}$ with elements $\left(d_{ij} = 1 + c_{ij}^2\right)$ is expressed with two unknown functions $c_0(\widetilde{\ell}_1)$ and $c_1(\widetilde{\ell}_1)$:

$$\mathbf{C} = \left\{ \begin{matrix} \alpha_1 c_0 & -\alpha_0 c_0 \\ -\alpha_1 c_1 & \alpha_0 c_1 \end{matrix} \right\}; \quad \mathbf{D} = \left\{ \begin{matrix} 1 + (\alpha_1 c_0)^2 & 1 + (\alpha_0 c_0)^2 \\ 1 + (\alpha_1 c_1)^2 & 1 + (\alpha_0 c_1)^2 \end{matrix} \right\} \tag{17}$$

the growth probabilities are then found from (15) :

$$\vec{p}(\widetilde{\ell}_1) = \left\{ \begin{matrix} p_0(\widetilde{\ell}_1) \\ p_1(\widetilde{\ell}_1) \end{matrix} \right\} = \frac{1}{\sqrt{1 + \alpha_0 \alpha_1 {c_0}^2} + \sqrt{1 + \alpha_0 \alpha_1 {c_1}^2}} \left\{ \begin{matrix} \sqrt{1 + \alpha_0 \alpha_1 {c_1}^2} \\ \sqrt{1 + \alpha_0 \alpha_1 {c_0}^2} \end{matrix} \right\} \tag{18}$$

4.1 Initial conditions

Fixing arbitrarily $\theta_0(0) = 0$, the initial conditions ($t = 0$), are:

$$\vec{L}(0) = \left\{ \begin{matrix} 0 \\ 0 \end{matrix} \right\}; \quad \left\{ \begin{matrix} \theta_0(0) \\ \theta_1(0) \end{matrix} \right\} = \left\{ \begin{matrix} 0 \\ \pi \end{matrix} \right\}; \quad \left\{ \begin{matrix} \phi_0(0) \\ \phi_1(0) \end{matrix} \right\} = \left\{ \begin{matrix} 2\operatorname{arccot}\sqrt{\alpha_1/\alpha_0} \\ -2\operatorname{arccot}\sqrt{\alpha_1/\alpha_0} \end{matrix} \right\} \tag{19}$$

and for parameter A,

$$A = (\alpha_0)^{-\alpha_0/2} (\alpha_1)^{-\alpha_1/2} /2. \tag{20}$$

From this we can calculate $\mathbf{C}(0)$ and $\mathbf{D}(0)$.

4.2 Equations satisfied by $c_0(\widetilde{\ell}_1)$ and $c_1(\widetilde{\ell}_1)$

From above, we deduce that the functions $c_0(\widetilde{\ell}_1)$ and $c_1(\widetilde{\ell}_1)$ verify the differential equation:

$$\frac{dc_0(\widetilde{\ell}_1)}{d\widetilde{\ell}_1} + \frac{\left(1+\alpha_0^2 c_0(\widetilde{\ell}_1)^2\right)\left(1+\alpha_1^2 c_0(\widetilde{\ell}_1)^2\right)}{\alpha_0\alpha_1\left(1+\alpha_0\alpha_1 c_0(\widetilde{\ell}_1)^2\right)\left(c_0(\widetilde{\ell}_1)+c_1(\widetilde{\ell}_1)\right)} = 0 \tag{21}$$

and, permuting c_0 and c_1, a similar equation for $c_1(\widetilde{\ell}_1)$. The solutions of these equations can be obtained parametrically:

$$c_0(\widetilde{\ell}_1) = 1/(u\sqrt{\alpha_0\alpha_1});\ c_1(\widetilde{\ell}_1) = u/\sqrt{\alpha_0\alpha_1} \tag{22}$$

with the relation,

$$\exp\left[\widetilde{\ell}_1\right] = u\left(\frac{\alpha_0+\alpha_1 u^2}{\alpha_1+\alpha_0 u^2}\right)^{(\alpha_0-\alpha_1)/4}. \tag{23}$$

The growth probabilities are then, from (18),

$$p_0(\widetilde{\ell}_1) = u/(1+u);\ p_1(\widetilde{\ell}_1) = 1/(1+u). \tag{24}$$

4.3 Discrete Fokker-Planck equation

The Fokker-Planck equation of a pair of needles is, with $\delta\widetilde{\ell}_1 = \delta\ell/2$,

$$P(\widetilde{\ell}_1, t+\delta t) = p_0\left(\widetilde{\ell}_1 - \delta\widetilde{\ell}_1\right)P(\widetilde{\ell}_1 - \delta\widetilde{\ell}_1, t) + p_1\left(\widetilde{\ell}_1 + \delta\widetilde{\ell}_1\right)P(\widetilde{\ell}_1 + \delta\widetilde{\ell}_1, t). \tag{25}$$

The evolution Eq. (25) can be expanded to second order in $\delta\ell$,

$$\partial P(\widetilde{\ell}_1, t)/\partial t = D.\partial^2_{\widetilde{\ell}_1{}^2}P(\widetilde{\ell}_1, t) - v.\partial_{\widetilde{\ell}_1}\left(P(\widetilde{\ell}_1, t)\,\mathcal{U}(\widetilde{\ell}_1)\right), \tag{26}$$

where the following constants have been introduced,

$$v = \delta\ell/(2\delta t) \text{ and } D = \delta\ell^2/(8\delta t), \tag{27}$$

v is the relative growth velocity of needle 0, D is the "diffusion" coefficient of the sticking between the two needles, and $\widetilde{\ell}_1 = (\ell_0 - \ell_1)/2$. At short time the particles stick at random on both needle tips, up to the moment when one needle gives way to the other. The function $\mathcal{U}$ characterizes the screening effect,

$$\mathcal{U}(\widetilde{\ell}_1) = \left(u(\widetilde{\ell}_1) - 1\right)/\left(u(\widetilde{\ell}_1) + 1\right), \tag{28}$$

where $u(\widetilde{\ell}_1)$ is implicitly defined by (23). The graph of $\mathcal{U}(\widetilde{\ell}_1)$ is shown in Fig.2a.

In the linear region, when $\widetilde{\ell}_1 \to 0$,

$$\mathcal{U}(\widetilde{\ell}_1) \sim \widetilde{\ell}_1/(2\alpha_0\alpha_1) \tag{29}$$

and the diffusion dominates, while when $\widetilde{\ell}_1 \to \pm\infty$, $\mathcal{U}(\widetilde{\ell}_1) \to \pm 1$, and the longest needle grows at velocity v. Eq. (26) represents the diffusion of a "particle" with coordinate $\widetilde{\ell}_1$ in a potential,

$$V(\widetilde{\ell}_1) = -v\int \mathcal{U}(\widetilde{\ell}_1)d\widetilde{\ell}_1. \tag{30}$$

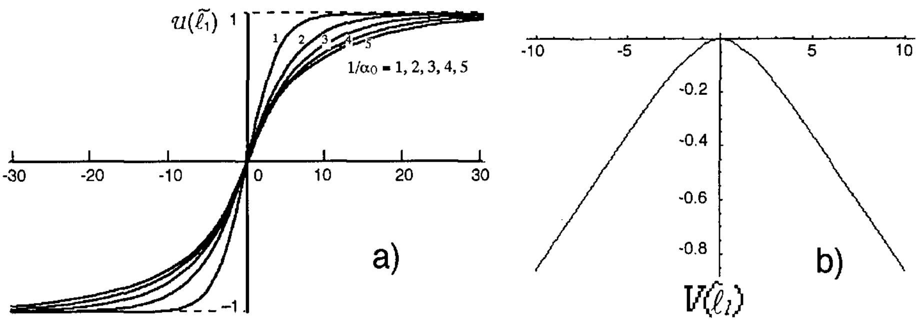

Figure 2. a) Graph of $\mathcal{U}(\widetilde{\ell}_1)$ for various values of α_0; b) Effective potential $V(\widetilde{\ell}_1)$ when $\alpha_0 = \alpha_1$.

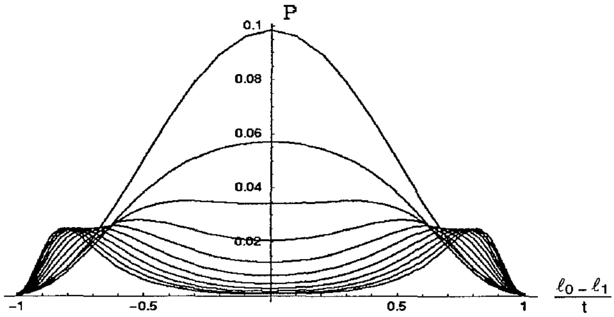

Figure 3. Distribution of probability $P(\widetilde{\ell}_1, t)$ at times $t/\delta t = 20, 30, .., 120$.

If the needles are equidistant, $\alpha_0 = \alpha_1 = 1$, and $u(\widetilde{\ell}_1) = \exp \widetilde{\ell}_1$, we have explicitly, $\mathcal{U}(\widetilde{\ell}_1) = \tanh\left(\widetilde{\ell}_1/2\right)$. The potential (shown in Fig.2b) is then explicitly,

$$V(\widetilde{\ell}_1) = -v \log\left(\cosh(\widetilde{\ell}_1/2)\right). \tag{31}$$

The time evolution of the solution of Eq. (26) for equal length needles ($\alpha_0 = \alpha_1 = 1$) at $t = 0$, is shown in Fig. 3.

4.4 Fluctuations of the difference between the branch lengths

The average of the difference between the branch lengths is zero by symmetry,

$$\left\langle \widetilde{\ell}_1 \right\rangle (t) = \int_{-\infty}^{+\infty} \widetilde{\ell}_1 P(\widetilde{\ell}_1, t) d\widetilde{\ell}_1 = 0 \tag{32}$$

because $\widetilde{\ell}_1 P(\widetilde{\ell}_1, t)$ is odd. But the mean square fluctuation

$$\left\langle \left(\widetilde{\ell}_1 - \left\langle \widetilde{\ell}_1 \right\rangle\right)^2 \right\rangle (t) = \left\langle \widetilde{\ell}_1^2 \right\rangle (t) = \int_{-\infty}^{+\infty} \widetilde{\ell}_1^2 P(\widetilde{\ell}_1, t) d\widetilde{\ell}_1 \tag{33}$$

is different from zero. Its time evolution is given by the partial differential equation,

$$\frac{\partial\left\langle\widetilde{\ell}_1^2\right\rangle}{\partial t}=\int_{-\infty}^{+\infty}\widetilde{\ell}_1^2\frac{\partial P(\widetilde{\ell}_1,t)}{\partial t}d\widetilde{\ell}_1= \tag{34}$$

$$\int_{-\infty}^{+\infty}\widetilde{\ell}_1^2\left(D.\partial^2_{\widetilde{\ell}_1{}^2}P(\widetilde{\ell}_1,t)-v.\partial_{\widetilde{\ell}_1}\left(P(\widetilde{\ell}_1,t)\,\mathcal{U}(\widetilde{\ell}_1)\right)\right)d\widetilde{\ell}_1.$$

Integrating by parts leads to,

$$\frac{\partial\left\langle\widetilde{\ell}_1^2\right\rangle}{\partial t}=2D+2v\left\langle\widetilde{\ell}_1\mathcal{U}(\widetilde{\ell}_1)\right\rangle. \tag{35}$$

When $\widetilde{\ell}_1\ll 1$, Eq. (35) with (29) reduces to,

$$\frac{\partial\left\langle\widetilde{\ell}_1^2\right\rangle}{\partial t}\approx 2D+\frac{v}{\alpha_0\alpha_1}\left\langle\widetilde{\ell}_1^2\right\rangle, \tag{36}$$

which has the following solution $\left(\left\langle\widetilde{\ell}_1^2\right\rangle(0)=0\right)$at small enough $\widetilde{\ell}_1$ or t,

$$\left\langle\widetilde{\ell}_1^2\right\rangle\approx\frac{2\alpha_0\alpha_1 D}{v}\left(\exp\left(\frac{vt}{\alpha_0\alpha_1}\right)-1\right), \tag{37}$$

The growth behaviour is diffusive $\left(\left\langle\widetilde{\ell}_1^2\right\rangle\approx 2Dt\right)$, up to a time $t_{dif}\simeq(\alpha_0\alpha_1/v)$.

5 Mullins-Sekerka instability of a comb of needles

Let us consider now the case of n equidistant needles for which $\overrightarrow{\alpha}=\frac{2}{n}\ {}^t\{1,1,...,1\}$. In this case it is convenient to use the Fourier transform of the above equations.

5.1 Reduction of the vector space dimension

We introduce the Fourier transform of an arbitrary n-vector $\overrightarrow{V}$:

$$\overrightarrow{\widetilde{V}}=\mathbf{F}.\overrightarrow{V}\quad\text{with}\quad\mathbf{F}=\{(1/n)\exp\left(2\pi\mathbf{i}\;jh/n\right)\}_{0\le j,h\le n-1} \tag{38}$$

To invert Eq. (12), we have to discard the completely symmetrical component, which makes $\mathbf{C}$ singular. The working space will have now $(n-1)$ components, and we will use an index r to specify its reduced vectors and the rectangular matrix operators relating reduced $(n-1)$-vectors to n-vectors. Thus, we introduce the rectangular Fourier matrix :

$$\mathbf{F}_r=\{F_{kh}\}_{1\le k\le n-1,\,0\le h\le n-1}\,. \tag{39}$$

The corresponding distribution of modes is,

$$\overrightarrow{\widetilde{L}}_r=\mathbf{F}_r.\overrightarrow{L}\,, \tag{40}$$

a particular mode k, being given by,

$$\widetilde{\ell}_k=(1/n)\sum_{j=0}^{n-1}\ell_j\exp\left(2\pi\mathbf{i}\;jk/n\right). \tag{41}$$

For an increase of length $\delta\ell$ on needle j,

$$\overrightarrow{\overrightarrow{\delta L_r}}[j] = \mathbf{F}_r.\overrightarrow{\delta L}[j] \; = \delta\ell/n \; \{\exp(2\pi i \; jk/n)\}_{1\leq k\leq n-1} \tag{42}$$

The reduced Fourier matrices lead to the following products: $\mathbf{F}_r.(\mathbf{F}^{-1})_r = \mathbf{1}_r$ and $(\mathbf{F}^{-1})_r.\mathbf{F}_r = \mathbf{1} - (1/n)\mathbf{O}$. In particular, this allows to write (12) as

$$\overrightarrow{\delta L} = -\frac{1}{n} \; (\mathbf{F}^{-1})_r.\mathbf{F}_r. \; \mathbf{C}.\overrightarrow{\delta\theta}. \tag{43}$$

It is now easy to invert Eq. (43). Let $\widetilde{\mathbf{C}}_r = \mathbf{F}_r.\mathbf{C}.(\mathbf{F}^{-1})_r$ then,

$$\overrightarrow{\delta\theta} = -n \; (\mathbf{F}^{-1})_r.(\widetilde{\mathbf{C}}_r)^{-1}\mathbf{F}_r.\overrightarrow{\delta L}. \tag{44}$$

5.2 Evolution of the probabilities

The probability $P(\overrightarrow{L}, t)$ to find the distribution of needle lengths $\overrightarrow{L}$ at time t is now replaced by the probability $P(\overrightarrow{\overrightarrow{L_r}}, t)$ to find a given Fourier mode, and we have to build the Fokker-Planck equation which relates $P(\overrightarrow{\overrightarrow{L_r}}, t)$ and $P(\overrightarrow{\overrightarrow{L_r}} + \overrightarrow{\overrightarrow{\delta L_r}}, t + \delta t)$.

With probability $p_i\left(\overrightarrow{\overrightarrow{L_r}}\right)$ we add at time t a particle of size $\delta\ell$ on needle i. The corresponding change of the lengths $\overrightarrow{\delta L}[i]$ induces a change in the angles θ and φ,

$$\overrightarrow{\delta\theta}[i] = -n \; (\mathbf{F}^{-1})_r.(\widetilde{\mathbf{C}}_r)^{-1}\mathbf{F}_r.\overrightarrow{\delta L}[i] \; ; \; \; \overrightarrow{\delta\varphi}[i] = \left\{\left\{\mathbf{D}.\overrightarrow{\delta\theta}[i]\right\}_h \left(\{\mathbf{D}.\overrightarrow{1}\}_h\right)^{-1}\right\} \tag{45}$$

from which we determine, by differentiation of Eq. (15), the value of $p_i(t + \delta t)$:

$$p_i(t+\delta t) = p_i(t) - \frac{\delta\Pi_i}{2\Pi_i^{3/2}\sum_j \Pi_j^{-1/2}} + \Pi_i^{-1/2}\Big(\sum_j \Pi_j^{-1/2}\Big)^{-2}\sum_h \frac{\delta\Pi_h}{2\Pi_h^{3/2}}. \tag{46}$$

5.3 Initial conditions

In the present case for which the needles have an equal length at $t = 0$, the initial angles ϕ_i et θ_j $(0 \leq i, j \leq n-1)$ are also regularly spaced :

$$\phi_i(0) = 2\pi i/n \;\; et \;\; \theta_j(0) = \pi(2j-1)/n. \tag{47}$$

5.4 Sticking probabilities in the linearized regime

From expression (45) we can calculate $\overrightarrow{\delta\theta}(0)$ and $\overrightarrow{\delta\varphi}(0)$ corresponding to a variation $\overrightarrow{\delta L}(0)$, for instance by adding a particle on needle h. Thus, if a particle of size $\delta\ell$ is stuck on needle h of a uniform comb, then the sticking probability of the next particle is, using Eq. (46), $(p(t=0) = \frac{1}{n}, D_j = \left\{\mathbf{D}.\overrightarrow{1}\right\}_j$, and $\sum_j D_j^{-1/2} = 1)$

$$p_i = \tfrac{1}{n} - \tfrac{\delta D_i(0)}{2n^3} + \tfrac{1}{2n^4}\textstyle\sum_{i,j}\delta\mathbf{D}(0)_{i,j}. \tag{48}$$

The sticking probability $p_i[h]$ of a new particle on needle i, while a particle is already stuck on needle h (δ_{ih} is the Kronecker symbol) is then,

$$p_i[h] = \frac{\delta_{ih}}{n} + \delta\ell\left(\frac{(n+1)(n-1)}{6n^2}\delta_{ih} - \frac{\csc^2((i-h)\pi/n)}{2n^2}(1-\delta_{ih})\right). \tag{49}$$

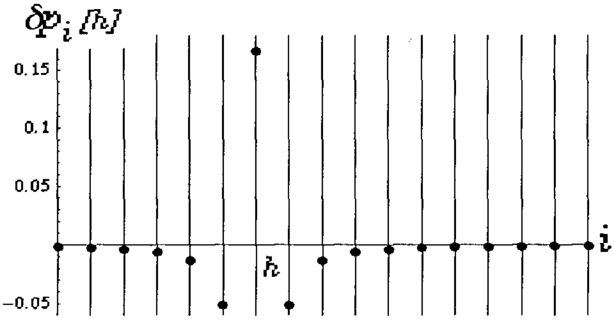

Figure 4. Screening of the sticking probability by the longer needle h.

As expected, in this result we see that the other needles are screened by the longer needle h. This is shown in $Fig.4$

6 Fokker-Planck equation of the linearized comb

Let $P_{lin}(\overrightarrow{L},t)$ be the linearized probability to find a set of needles of size $\overrightarrow{L} = {}^t\{\ell_0,\ell_1,...,\ell_h,...,\ell_{n-1}\}$; the linearization supposes that the lengths ℓ_i are not too different in such a way that no needle can be completely screened (see Sec. 4). In the linearized approximation, the sticking probabilities on $\overrightarrow{L}$ are then the superposition of the individual probabilities $p^j[\ell_h]$, where $p^j[\ell_h]$ is deduced from $p_j[h]$ by replacing $\delta\ell$ by ℓ_h (superposition rule). If between t and $t+\delta t$, a particle $\delta\ell$ is added on needle h and if $\overrightarrow{L'_h} = \overrightarrow{L} - \overrightarrow{\delta L}[h]$, then

$$P_{lin}(\overrightarrow{L},t+\delta t) = \sum_{h=0}^{n-1}\sum_{j=0}^{n-1} p^h[\ell'_j] \;\; P_{lin}(\overrightarrow{L'}[h],t), \tag{50}$$

with $\overrightarrow{L'}[h] = {}^t\{\ell_0+\frac{\delta\ell}{n},\ell_1+\frac{\delta\ell}{n},...,\ell_h-\delta\ell+\frac{\delta\ell}{n},...,\ell_{n-1}+\frac{\delta\ell}{n}\}$.

We recover the linearized form of the particular case $n=2$ of (25):

$$P_{lin}(\widetilde{\ell}_1,t+\delta t) = \left(p^0[\ell'_0]+p^0[\ell'_1]\right) P_{lin}(\widetilde{\ell}_1-\delta\widetilde{\ell}_1,t) + \left(p^1[\ell'_0]+p^1[\ell'_1]\right) P_{lin}(\widetilde{\ell}_1+\delta\widetilde{\ell}_1,t) \tag{51}$$

as to first order,

$$\begin{aligned} p_0\left(\widetilde{\ell}_1-\delta\widetilde{\ell}_1\right) &\simeq \left(1+(\ell_0-\ell_1-\delta\ell)/4\right)/2 = p^0[\ell'_0]+p^0[\ell'_1] \\ p_1\left(\widetilde{\ell}_1+\delta\widetilde{\ell}_1\right) &\simeq \left(1-(\ell_0-\ell_1-\delta\ell)/4\right)/2 = p^1[\ell'_0]+p^1[\ell'_1]. \end{aligned} \tag{52}$$

In the general linearized case we obtain from Eqs. (49,50),

$$\begin{aligned} P_{lin}(\overrightarrow{L},t+\delta t) &= \tfrac{1}{n}\textstyle\sum_{h=0}^{n-1} P_{lin}(\overrightarrow{L'}[h],t) + \\ &\tfrac{1}{2n^2}\textstyle\sum_{h=0}^{n-1}\sum_{j=0,j\neq h}^{n-1}\left(\ell'_h-\ell'_j\right)\csc^2\left((h-j)\pi/n\right) P_{lin}(\overrightarrow{L'}[h],t) \end{aligned} \tag{53}$$

with $\overrightarrow{\widetilde{L}'_r}[h] = \overrightarrow{\widetilde{L}'_r} - \overrightarrow{\delta\widetilde{L}_r}[h]$ defined by Eq. (42).

6.1 *Fourier transform of the kinetic equation, and the Fokker-Planck equation*

Equation (53) can be written after Fourier transformation,

$$P_{lin}(\overrightarrow{\widetilde{L}}_r, t+\delta t) = \sum_{h=0}^{n-1}\left(\tfrac{1}{n} - \tfrac{n^2-1}{6n^2}\delta\ell + \sum_{k=1}^{n-1} e^{-\frac{2\pi i}{n}hk}\tfrac{k}{n}\left(1-\tfrac{k}{n}\right)\widetilde{\ell}_k\right) P_{lin}(\overrightarrow{\widetilde{L}}_r - \overrightarrow{\delta\widetilde{L}_r}[h], t). \tag{54}$$

In the above equation, we have used the relation,

$$\sum\nolimits_{j\neq 0} e^{-\frac{2\pi i}{n}jk}\csc^2(j\pi/n) = (n^2 - 6kn + 6k^2 - 1)/3. \tag{55}$$

Second order Taylor expansion of the above Eq. (54) with respect to $\overrightarrow{\delta\widetilde{L}_r}[h]$ makes it clearer. Neglecting the third order terms in $\delta\ell$, we find the remarkable Fokker-Planck equation (note that $\widetilde{\ell}_{-k} \equiv \widetilde{\ell}_{n-k}$, k being defined modulo n),

$$\frac{\partial P_{lin}(\overrightarrow{\widetilde{L}}_r, t)}{\partial t} = -\frac{\delta\ell}{\delta t}\sum_{k=1}^{n-1}\frac{\partial}{\partial\widetilde{\ell}_k}\left[\frac{k}{n}\left(1-\frac{k}{n}\right)\widetilde{\ell}_k\, P_{lin}(\overrightarrow{\widetilde{L}}_r, t)\right]$$
$$+\frac{\delta\ell^2}{2n^2\delta t}\sum_{k'=1}^{n-1}\frac{\partial^2 P_{lin}(\overrightarrow{\widetilde{L}}_r, t)}{\partial\widetilde{\ell}_{k'}\partial\widetilde{\ell}_{-k'}} + \frac{\delta\ell^2}{2n\delta t}\sum_{k,k'=1}^{n-1}\frac{\partial^2 P_{lin}(\overrightarrow{\widetilde{L}}_r, t)}{\partial\widetilde{\ell}_{k'}\partial\widetilde{\ell}_{k-k'}}. \tag{56}$$

When $n = 2$, we recover (26) linearized with $\alpha_0 = \alpha_1 = 1$, and $\mathcal{U}(\widetilde{\ell}_1) \sim \widetilde{\ell}_1/2$ from Eq. (29).

6.2 *Short discussion of the linearized kinetic equation*

Equation (56) contains three terms :

i) the first term is a drift term, associated with the growth velocity v. At the end of the linear regime, we expect an exponential screening behavior similar to $\mathcal{U}(\widetilde{\ell}_1)$ in the 2-needles case. The remarkable point lies in the factor $\frac{k}{n}\left(1-\frac{k}{n}\right)\widetilde{\ell}_k$. If we start from a situation where all the modes k have the same weight $\widetilde{\ell}_k \equiv \widetilde{\ell}$, then due to the $\frac{k}{n}\left(1-\frac{k}{n}\right)$ coefficient, the mode $k_{\max} = n/2$ dominates progressively since it has the highest growth rate. Due to screening, most of the other modes disappear, and the system becomes equivalent to a comb of $n/2$ needles: a succession of period doubling is the result of the growth.

ii) The second term couples the mode k and $n-k$.

iii) The third term is a mode coupling term, which merges two modes k' and k'' into a mode $k = k' + k''$.

6.3 *Correlation between modes and fluctuations of a mode q, in the initial regime*

We first calculate the correlation between modes,

$$\left\langle\widetilde{\ell}_{q_1}\widetilde{\ell}_{q_2}\right\rangle(t) = \int_{-\infty}^{+\infty}\widetilde{\ell}_{q_1}\widetilde{\ell}_{q_2}P(\overrightarrow{\widetilde{L}}_r, t)d\overrightarrow{\widetilde{L}}_r \quad . \tag{57}$$

Integration by parts using Eq. (56) gives,

$$\partial_t \left\langle \widetilde{\ell}_{q_1} \widetilde{\ell}_{q_2} \right\rangle_{lin}\Big|_{q_1 \neq q_2} = \frac{\delta \ell}{\delta t}\left(\frac{q_1}{n}\left(1-\frac{q_1}{n}\right)+\frac{q_2}{n}\left(1-\frac{q_2}{n}\right)\right)\left\langle \widetilde{\ell}_{q_1} \widetilde{\ell}_{q_2} \right\rangle + \\ + \frac{\delta \ell^2}{2n^2 \delta t}\delta_{q_1+q_2,0} + \frac{\delta \ell^2}{2n\delta t}\left(1-\delta_{q_1+q_2,0}\right). \tag{58}$$

The fluctuation of a mode q, can be also determined from Eq. (56) and leads to,

$$\partial_t \left\langle \widetilde{\ell}_q^{\,2} \right\rangle_{lin} = \frac{2\delta \ell}{\delta t}\frac{q}{n}\left(1-\frac{q}{n}\right)\left\langle \widetilde{\ell}_q^{\,2} \right\rangle + \\ + \frac{\delta \ell^2}{n^2 \delta t}\delta_{q,n/2} + \frac{\delta \ell^2}{n\delta t}\left(1-\delta_{q,n/2}\right), \tag{59}$$

which has a solution similar to the two-needles case, which underlined the existence of a q-dependent effective velocity,

$$v_{eff}(q) = \frac{2\delta \ell}{\delta t}\frac{q}{n}\left(1-\frac{q}{n}\right) \tag{60}$$

$$\left\langle \widetilde{\ell}_q^2 \right\rangle_{lin} = \frac{2D_1\delta_{q,n/2} + 2D_2\left(1-\delta_{q,n/2}\right)}{v_{eff}(q)}\left[\exp\left(v_{eff}(q)t\right)-1\right] \tag{61}$$

and shows that the mode $q = n/2$ has the fastest growth. We have noted,

$$D_1 = \frac{\delta \ell^2}{2n^2\delta t} \quad \text{and} \quad D_2 = \frac{\delta \ell^2}{2n\delta t}. \tag{62}$$

The correlation between the modes has a similar behaviour,

$$\left\langle \widetilde{\ell}_{q_1} \widetilde{\ell}_{q_2} \right\rangle_{lin} = \frac{D_1\delta_{q_1+q_2,0} + D_2\left(1-\delta_{q_1+q_2,0}\right)}{v_{eff}(q_1,q_2)}\left[\exp\left(v_{eff}(q_1,q_2)t\right)-1\right] \tag{63}$$

with an effective velocity,

$$v_{eff}(q_1,q_2) = \frac{\delta \ell}{\delta t}\left(\frac{q_1}{n}\left(1-\frac{q_1}{n}\right)+\frac{q_2}{n}\left(1-\frac{q_2}{n}\right)\right) \tag{64}$$

Further studies are in progress to relate the linear regime to the screened regime and to obtain analytically the scaling laws suggested numerically by Krug *et al.*[4] and Adda Bedia[14].

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